Dynamical Systems Model and Discrete Element Simulations of a Tapped Granular Column

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Abstract. We present an approximate dynamical systems model for the mass center trajectory of a tapped column of N uniform, inelastic, spheres (diameter d), in which collisional energy loss is governed by the Walton-Braun linear loading-unloading soft interaction. Rigorous analysis of the model, akin to the equations for the motion of a single bouncing ball on a vibrating plate, reveals a parameter $\gamma := 2a\omega^2(1+e)/g$ that gauges the dynamical regimes and their transitions. In particular, we find bifurcations from periodic to chaotic dynamics that depend on frequency ω, amplitude a/d of the tap. Dynamics predicted by the model are also qualitatively observed in discrete element simulations carried out over a broad range of the tap parameters.

Keywords: Dynamical systems, tapped column, DEM simulations.

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INTRODUCTION

A tapped assembly of granular materials will exhibit a variety of behaviors that depend on the tap amplitude a, frequency ω and the duration of the relaxation interval between taps τ_r. Over the last few decades, there have been numerous reports in the literature on this topic (mostly for $\tau_r = 0$) focused on wave motion, surface behavior, fluidization, density relaxation and segregation phenomena (e.g., [1-11]). Perhaps the most ostensibly clear-cut problem is the one-dimensional analog, i.e., a column of N uniform spheres. The simplest case of a single ball on a continuously oscillating plate is well-understood (albeit complex [12]) and is often regarded as a paradigm dynamical system. This paper is focused on understanding the dynamics of a column of uniform spheres that are subjected to taps imposed by the motion of a rigid floor. Dynamics are studied over time scales that are large as compared with the compression-expansion wave that propagates through the system [13]. We consider the efficacy of the mass center to capture the overall behavior of entire stack of spheres – cognizant of actual complexities of the individual dynamics of the particles comprising the column. More specifically, our investigation addresses the ability of the mass center to qualitatively predict essential features of the column dynamics (e.g., period-doubling bifurcations and transitions to chaos).

DYNAMICAL SYSTEMS MODEL

We apply Newton's laws to a stack or column of spheres (diameter d, mass density ρ) under gravity whose collisional exchanges are governed by a Walton-Braun [14] type linear loading/unloading interaction. The role of inelasticity is characterized by restitution coefficient (e) that quantifies energy loss through particle-particle and particle-floor collisions. The resulting system of N nonlinear, second-order ordinary differential equations for the particle centers locations y_i is reduced to a single equation for the mass center via a plausible assumption on the relationship between $\bar{y}(t)$ and $y_1(t)$. A sketch of the derivation follows. (See [15] for details).

Derivation of the Model

Let $0 \le y_0 < y_1 < \cdots < y_N$ be the positions of the centers of the spheres of radius $r = d/2$, such that the floor location y_0 is a periodic function given by

$$y_0(t) = \begin{cases} a\sin(\omega t), & 0 \le t \le \pi/\omega \\ 0, & \pi/\omega \le t \le T \end{cases} \quad (1)$$

where $0 \le t \le T$, $\pi/\omega \ll T$, and $a > 0$. According to Newton's law, for $1 \le i \le N$, we have

$$\ddot{y}_i = -g + m^{-1}\left(f_i^{i-1} + f_i^{i+1}\right) \quad (2)$$

In (2), f_i^{i-1}, f_i^{i+1} are the forces on particle i imparted by $i-1$ and $i+1$, respectively ($1 \le i \le N-1$):

$$f_1^0 = \kappa[1 - \varepsilon\hbar(\Delta\dot{y}_0)]\alpha_0\chi(\alpha_0)\mathcal{F}(|\Delta y_0|, \Delta\dot{y}_0) \quad (3)$$

$$\Delta\dot{y}_0 := \dot{y}_1 - \dot{y}_0\,;\ \alpha_0 := r - \Delta y_0;\ \Delta y_0 := (y_1 - y_0)$$

$$f_i^{i+1} = -\kappa[1 - \varepsilon\hbar(\Delta\dot{y}_i)]\alpha_i\chi(\alpha_i)\mathcal{F}(|\Delta y_i|, \Delta\dot{y}_i) \quad (4)$$

$$\Delta\dot{y}_i := \dot{y}_{i+1} - \dot{y}_i; \alpha_i := d - \Delta y_i; \Delta y_i := y_{i+1} - y_i$$

$$f_N^{N+1} = 0 \quad (5)$$

$$f_i^{i-1} = -f_{i-1}^i \quad (6)$$

where,

$$\hbar(q) := \begin{cases} -1, & q < 0 \\ 0, & q = 0 \\ 1, & q > 0 \end{cases}, \quad \chi(s) := \begin{cases} 0, & s \le 0 \\ 1, & s > 0 \end{cases} \quad (7)$$

and $\mathcal{F}(u,v)\colon (0,\infty) \mapsto [1,\infty)$ is a continuous penalty function designed to ensure that particles cannot pass though one another or the floor. We note that $\mathcal{F}$ is only active at large overlaps α when particles are approaching each other. Initial locations of the spheres are determined by setting $\dot{y}_i = \ddot{y}_i = 0$ $(0 \le i \le N)$ in (2), resulting in a system of linear equations that are solved for $\{y_1(0), \ldots, y_N(0)\}$. In (3) and (4), the loading and unloading springs have stiffness values $\kappa(1+\varepsilon)$ and $\kappa(1-\varepsilon)$, respectively. This results in a restitution coefficient $e = (1-\varepsilon^2)/(1+\varepsilon^2)$, with $e = 0$ ($e = 1$) corresponding to the limit of perfectly plastic (elastic) collisions.

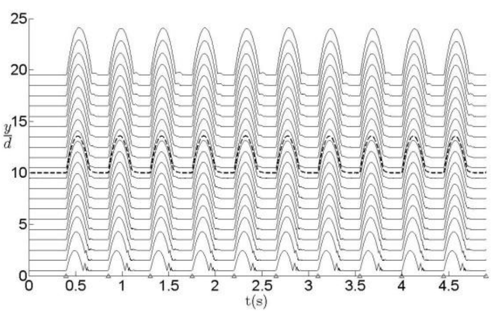

FIGURE 1. Simulated trajectories of the mass center (bold dotted line) and spheres over 10 taps at $f = 15\ Hz$ and $a/d = 0.5$. Triangles on the t-axis represent tap initiation.

Reduced Mass Center Model

The equation for the mass center $\bar{y} := \frac{1}{N}\sum_{i=1}^{N} y_i$, found directly by summing (2) over the index i is,

$$\ddot{\bar{y}} = -g + \frac{\kappa}{M}[1 - \varepsilon\hbar(\Delta\dot{y}_0)]\alpha_0\chi(\alpha_0)\mathcal{F}(|\Delta y_0|, \Delta\dot{y}_0) \quad (8)$$

$$\bar{y}(0) = N^{-1}\textstyle\sum_{k=1}^{N} y_k(0)\,,\ \dot{\bar{y}}(0) = 0 \quad (9)$$

Since the right-hand side of (8) contains $y_1(t)$, it is necessary to assume a relationship between the motion of the sphere adjacent to the floor and $\bar{y}(t)$. We postulate that $y_1(t) \approx \bar{y}(t)/N$; thus, the reduced mass center model with $z := \bar{y}$ and $z^* := \frac{z}{N} - y_0$ becomes,

$$\ddot{z} = -g + \frac{\kappa}{M}[1 + \varepsilon\hbar(\dot{z^*})](r - z^*) \chi(r - z^*)\mathcal{F}(|z^*|, \dot{z^*}) \quad (10)$$

$$z(0) = N^{-1}\textstyle\sum_{k=1}^{N} y_k(0)\,,\ \dot{z}(0) = 0 \quad (11)$$

Our assumption that $y_1(t) \approx \bar{y}(t)/N$ is reasonable when periodic behavior takes place (see Fig. 1); however, we find that other features of the dynamics of the mass center in (10) signal the same transitions that occur in the full system (2) – (7). We demonstrate this via discrete element simulations described in a subsequent section.

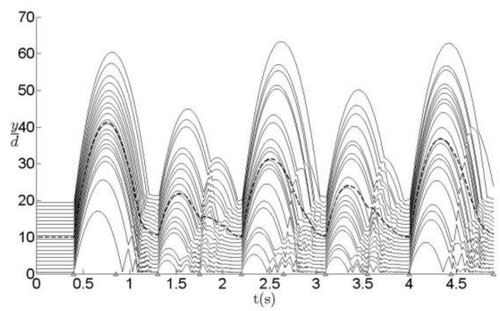

FIGURE 2. Simulated trajectories of the mass center $\bar{y}/d$ (bold dotted line) and spheres over 10 taps at $f = 15\ Hz$ and $a/d = 1.5$. ($e_p = e_f = 0.95$)

Correspondence with Bouncing Ball

It can be shown that the reduced model (10) and (11) is equivalent to that of a bouncing ball having mass NM, under gravity g/N, and restitution coefficient $e = (1-\varepsilon^2)/(1+\varepsilon^2)$. Consequently, all results available for the single ball apply to the mass center dynamics, and in particular, the existence of period doubling, chaotic regimes and strange attractors. Specifically, one can use an approximate discrete dynamical model similar to that employed by Holmes [16] to analyze the motion.

The idea is to monitor successive times $\{t_n\}$ of impact with the floor and the corresponding velocities $\{v_n\}$, which are directly related to the coefficient of restitution e, via a difference equation of the form

$$(t_{n+1}, V_{n+1}) = \Phi(t_n, V_n) \quad (12)$$

that can be taken to be defined on an infinite circular cylinder owing to the periodicity in the timing of the taps. More precisely, the system (12) is recast in non-dimensional form as

$$\begin{aligned} \theta_{n+1} &= \theta_n + v_n, \quad mod(\omega T) \\ v_{n+1} &= ev_n + \gamma W(\theta_n + v_n) \end{aligned} \quad (13)$$

where $mod(\omega T)$ means treats all values differing by an integral multiple of ωT as equal, $\theta := \omega t$, $v := 2\omega V/g$, $\gamma := 2a\omega^2(1+e)/g$ and W is an ωT-periodic function defined on a period interval as

$$W(s) := \begin{cases} \cos(s), & 0 \le s \le \pi \\ 0, & \pi \le s \le \omega T \end{cases} \tag{14}$$

Observe that the parameter γ is proportional to the acceleration $a\omega^2$ of the floor and that (13) is in the form of a "standard" map [16] with dynamics (iterates) that can be analyzed in depth.

For example, analysis of (13) in [15, 16] shows that, like the bouncing ball, there is a period-doubling cascade leading to chaos corresponding to increasing values of γ. In addition, full-fledged horseshoe type chaos, and strange attractors (inelastic collisions) can be shown to exist for certain sufficiently large values of γ. Among these and other facts to be inferred from the analysis is that there are arbitrarily large parameter windows for which (13) has stable cycles of period three and higher odd periods.

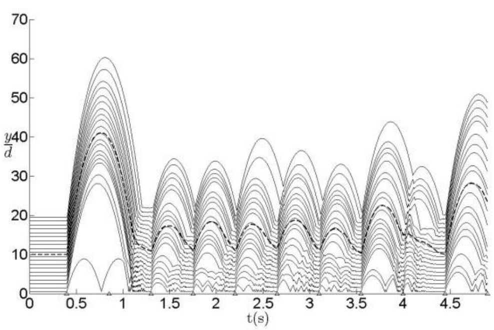

FIGURE 3. Simulated trajectories of the mass center $\bar{y}/d$ (bold dotted line) and spheres over 10 taps at $f = 15$ Hz and $a/d = 1.5$, $e_f = 1.0$, $e_p = 0.95$.

There is another discrete dynamical model that is directly related to the height y and velocity w of the mass center of the ball at the periodic times of application of the taps. This is manifested in the Poincaré map and represented by

$$(y_{n+1}, w_{n+1}) = S(y_n, w_n). \tag{15}$$

An analysis of (15) actually shows the same range of dynamical behaviors as (13).

One can also consider several discrete dynamical variations of the systems (13) and (15) – an example of which is currently being explored via simulation. Here, the height is fixed to be the radius of the ball and the successive times $\{\tau_n\}$ when the mass center is at that height are determined. The result is the one-dimensional discrete dynamical system modeled as

$$\tau_{n+1} = \Lambda(\tau_n) \tag{16}$$

A fixed point of Λ corresponds to a periodic motion of the ball, while a fixed point of Λ^2 that is not fixed by Λ represents period doubling. If no such pattern is discernible, chaos is strongly indicated. Several simulations of iterates of (16), which one can identify with the evolution of the phase of the motion, will be carried out and reported in a future paper.

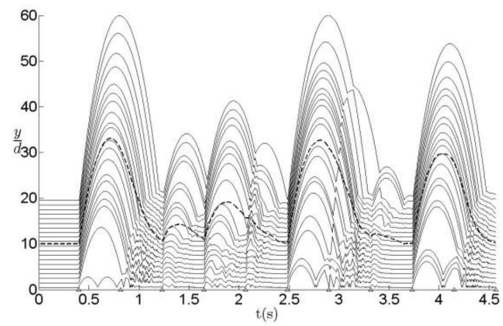

FIGURE 4. Solid lines are the sphere trajectories at $a/d = 1$ and $f = 30$ Hz, while the dotted line is the mass center trajectory.

DISCRETE ELEMENT SIMULATIONS

We considered a column of $N = 20$ uniform spheres initially resting on a floor which moves according to (1) with relaxation $\tau_r = 0.4\ s$ with $T := (\pi/\omega) + \tau_r$. Interactions between colliding spheres follows the Walton-Braun model [14] where loading and unloading is governed by linear springs of stiffness K_1 and K_2 (respectively), with $K_1 < K_2$ and $e = \sqrt{K_1/K_2}$. We selected an integration time step three orders of magnitude smaller than that given by the loading period $\sim\sqrt{m/K_1}$ in order to accurately capture the dynamics and minimize computational round-off errors. The restitution coefficient ($e_p = 0.95$) used in our studies was selected from experimentally reported [17] values for acrylic spheres.

We carried out the following parameter studies: (1) $f = 10$ Hz, $a/d = 0.25, 0.50, 0.75, 1.0, 1.25, 1.50$, and (2) $a/d = 0.50$, $f = 5, 10, 15, 20, 25, 30$ Hz. Periodic mass center trajectories that occurred were run for 100 taps to ensure that the dynamics remained stable. Fig. 1 shows the first 10 taps ($a/d = 0.50$, $f = 15$ Hz) of a periodic trajectory, while Fig. 2 shows the results for $a/d = 1.50$ and $f = 15$ Hz. Triangle markers placed on the t-axis indicate the times when taps were applied. A small change in the restitution coefficient of the floor e_f from 0.95 to 1.0 drastically modifies the column dynamics (Fig. 3), a sensitivity characteristic of chaos. Most noticeable is the difference in the trajectory of the particle nearest the floor at $e_f = 1$ that ultimately influences the dynamics of the other particles. However, at lower

a/d values where the mass center motion was periodic, this small change in e_f engendered no perceptible effect on the dynamics. Observations of the results showed that transitions of the column dynamics from periodic to chaotic behavior as γ increased were reflected in the mass center trajectories in accordance with our analysis of the model.

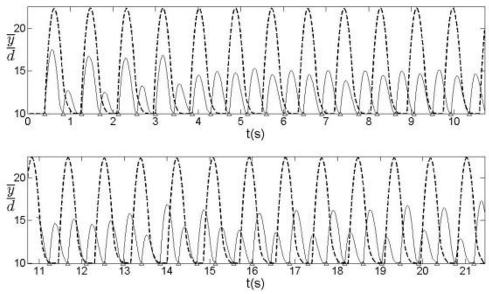

FIGURE 5. Trajectories $\bar{y}/d$ at $a/d = 1$ shown for $f = 15$ Hz (solid line) and $f = 20$ Hz (dotted line).

Variations in tap frequency at fixed a/d had an analogous effect on the system, i.e., larger frequencies resulted in chaotic trajectories. Chaotic trajectories at $f = 30$ and $a/d = 1$ of the mass center and column particles are presented in Fig. 4 over 10 taps. mass center at $f = 15$ Hz (solid line) and $f = 20$ Hz (dotted line) over 50 taps for $a/d = 1$, where one might expect chaotic instead of the periodicity observed. This seemingly anomalous behavior is suggested by our analysis ([15]) that revealed windows corresponding to large γ values where (13) has stable periodic orbits.

CONCLUSIONS

We derived an approximate dynamical system model for the behavior of the mass center of a column of uniform, inelastic spheres subjected to the tapping induced by a floor. Collisional exchanges between particles and particle-floor interactions obeyed the Walton-Braun linear loading-unloading law. Upon making a rather plausible assumption concerning the trajectory of the particle closest to the floor, it was shown that mass center dynamics, represented by a system of just two differential equations (reduced from 2*N*), is essentially equivalent to that of a single massive ball on a periodically tapped floor. The evolution of the reduced continuous system was further approximated by a discrete dynamical system (expressed as a pair of difference equations). Analysis of the discrete dynamical model revealed various behavioral regimes (e.g., periodicity, period doubling paths to chaos) characterized by a single bifurcation parameter $\gamma := a\omega^2(1+e)/g$, which correlated well with simulations of a column of 20 particles. We observed that the mass center trajectory appeared to be a good indicator of transitions from regular to chaotic dynamics of the column. Simulations also confirmed the effectiveness of the mass center in predicting the onset of chaos and the importance of γ. Current efforts involve computation of the Poincaré map and Lyapunov exponents via simulation over an expanded parameter space that includes the number of particles and relaxation time between taps. In addition, we are investigating the possibility of using similar methods to describe the evolution of particle densities manifested by the distances among spheres.

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